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MATHEMATICAL MODEL OF AN INCOMPRESSIBLE VISCOELASTIC MAXWELL MEDIUM

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Nonstationary motions of incompressible viscoelastic Maxwell continuum with a constant relaxation time are considered. Because in an incompressible continuous medium, pressure is not a thermodynamic variable but coincides with the stress-tensor trace to within a factor, it follows that, separating the spherical part from this tensor, one can assume that the remaining part of the stress tensor has zero trace. In the case of an incompressible medium, the equations for the velocity, pressure, and stress tensor form a closed system of first-order equations which has both real and complex characteristics, which complicates the formulation of the initial-boundary-value problem. Nevertheless, the resolvability of the Cauchy problem can be proved in the class of analytic functions. Unique resolvability of the linearized problem was established in the classes of functions of finite smoothness. The class of effectively one-dimensional motions for which the subsystem of three equations is a hyperbolic one was studied. The results of an asymptotic analysis of the latter imply the possible formation of discontinuities during the evolution of the solution. The general system of equations of motion admits an infinite-dimensional Lie pseudo-group which contains an extended Galilean group. The theorem of the invariance of the conditions on the a priori unknown free boundary was proved to obtain exact solutions of free-boundary problems. The problem of deformation of a viscoelastic strip subjected to tangential stresses applied to the free boundary is considered as an example of application of this theorem. In this problem, a scale effect of short-wave instability caused by the absence of diagonal dominance of the stress tensor deviator was found.

Key words: *viscoelastic medium, incompressibility, Maxwell relation, Galilean group, free-boundary problems.*

Introduction. The Maxwell viscoelastic model has been the subject of extensive mathematical studies [1–4]. This model is used to describe the behavior of metals under pulsed loading and the motion of melts and solutions of polymers. However, whereas in the first case, it is necessary to take into account the compressibility of the medium, in polymer flow, the compressibility factor does not play a significant role and the flow velocity field can therefore be considered solenoidal. The present paper deals with a study of the mathematical properties of incompressible viscoelastic Maxwell continuum. The material characteristics of this medium are its density ρ, dynamic viscosity μ, and relaxation time τ. We denote the velocity, pressure, stress tensor, and strain rate tensor by $\boldsymbol{v}$, p, P, and D, respectively. Next, it is assumed that the medium is not acted upon by volume forces or is acted upon by external forces which have a potential.

The sought functions are related by three equations: the scalar continuity equation, the vector equation of momentum, and the tensor rheological relation. The first two equations have universal form (see, for example, [5]), whereas in the choice of the last equation there is arbitrariness [1, 3]. The problem of a rational choice of the rheological relation for an incompressible viscoelastic Maxwell medium is considered in Sec. 2.

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1. Equations of Motion. The continuity equation for an incompressible continuous medium has the form

$$\operatorname{div} \boldsymbol{v} = 0. \tag{1.1}$$

The momentum equation for any continuous medium satisfying the Cauchy stress principle is written as

$$\rho(\boldsymbol{v}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{v}) = \operatorname{div} P. \tag{1.2}$$

To formulate the closing rheological relation, we write the stress tensor as

$$P = -pI + S, \tag{1.3}$$

where I is unit tensor. From the assumption of the incompressibility of the medium, it follows that the pressure is no longer a thermodynamic variable but arises as a constraint reaction of the mechanical system force to the kinematic constraint — solenoidailty of the velocity field. The physical meaning of the quantity p in an incompressible fluid is discussed in detail by Serrin [5], who used the notion of the mean pressure

$$\bar{p} = -(1/3) \operatorname{tr} P \tag{1.4}$$

and showed that the mean pressure $\bar{p}$ coincides with the usual pressure p only in the case of a linear relationship between the tensors D and P. Below, it is assumed that a linear relationship between these tensors also exists for relaxing media, in particular, for Maxwell continuum. In this case, by analogy with a Newtonian incompressible viscous fluid, we will identify the mean pressure with the usual pressure. Then, Eqs. (1.3) and (1.4) and the equality $p = \bar{p}$ lead to

$$\operatorname{tr} S = 0. \tag{1.5}$$

In other words, the tensor S is the deviator of the stress tensor P. Unlike in the case of a compressible Maxwell medium, where the relaxation relation is written for the entire tensor P [1–4], we will assume that this relation is satisfied only for the deviator part S of the tensor P:

$$\tau \frac{\tilde{d}S}{dt} + S = 2\mu D. \tag{1.6}$$

Here $\tilde{d}/dt$ is one of the invariant or objective derivatives of the tensor [1, 3]. The choice of this derivative is ambiguous: it can be the upper or lower convective derivative, the corotational Jaumann derivative or their combinations. It is important that, with this choice, relation (1.6) is invariant under rotation with arbitrary angular velocity.

The validity of the above assumption follows from the reasoning based on the results of [6, 7]. Apakshev and Pavlov [6] studied the inertial axial motion of a vertical cylinder placed in a vessel of large diameter filled with water. The viscosity effect was responsible for a reduction in the angular velocity of the cylinder with time, as was observed in the experiment. However, it turned out that in the final stage of rotation, the velocity of rotation of the cylinder changed nonmonotonically. Damping oscillations with a period of about one half of an hour were found. Korenchenko and Beskachko [7] investigated the inertial rotation of a solid disk placed on the free water surface in a cylindrical vessel and found a damping oscillation mode. Similar modes have also been observed in other working media. This suggests that, along with viscosity, elasticity is a significant factor which determines the behavior of water at low shear strain rate of the order of 10^{-3} sec^{-1}. In [6], the shear modulus of water G at low strain rates was estimated to be of the order of 10^{-6} N/m^2. A close estimate of G can be obtained by analyzing the results of [7]. At the same time, the bulk compression modulus of water $K \gg G$ can be estimated knowing the sound velocity, since $K = \rho c^2$.

Substitution of the value $c = 1500$ m/sec into the last formula yields $K = 2.25 \cdot 10^9$ N/m^2. In mechanics of viscoelastic media, it is assumed that the stress relaxation time is similar in order of magnitude to the ratio of the viscosity to the elastic modulus. Denoting the bulk compression stress relaxation time by τ^*, we obtain the estimate $\tau^* = \tau G K^{-1}$, where τ is the shear stress relaxation time. Setting $G = 1.5 \cdot 10^{-6}$ N/m^2 and $\tau = 1.5 \cdot 10^3$ sec, we have $\tau^* = 10^{-12}$ sec. In developing the model for the behavior of water and similar fluids under experimental conditions [6, 7], we assume that the compression stresses relax instantaneously. In an incompressible fluid, the spherical part of the stress tensor is determined by pressure. The aforesaid explains why relation (1.6) dose not contain pressure. There is reason to expect that the proposed model of a viscoelastic Maxwell medium can be used to study fluid motion under microgravity and microscale motions of fluids and to describe the final stage of the approach of fluids to equilibrium.

Let us consider relation (1.6), choosing as an invariant derivative the corotational Jaumann derivative:

$$\tau\Big(\frac{\partial S}{\partial t} + \boldsymbol{v}\cdot\nabla S - W\cdot S + S\cdot W\Big) + S = 2\mu D \tag{1.7}$$

(W is the antisymmetric part of the tensor $\nabla \boldsymbol{v}$). Then, for any symmetric tensor S, we have

$$\operatorname{tr}(S\cdot W - W\cdot S) = 0.$$

Bu virtue of Eqs. (1.1), $\operatorname{tr} D = 0$. This relation and equality (1.6) lead to the following equation for the function $\operatorname{tr} S$:

$$\tau\Big(\frac{\partial \operatorname{tr} S}{\partial t} + \boldsymbol{v}\cdot\nabla \operatorname{tr} S\Big) + \operatorname{tr} S = 0.$$

If $\operatorname{tr} S = 0$ at the time $t = 0$, then, in view of the last equation, $\operatorname{tr} S = 0$ for any t.

Let us write equality (1.6) with the upper convective derivative:

$$\tau\Big(\frac{\partial S}{\partial t} + \boldsymbol{v}\cdot\nabla S - \nabla\boldsymbol{v}\cdot S - S\cdot\nabla\boldsymbol{v}^{\mathrm{t}}\Big) + S = 2\mu D. \tag{1.8}$$

Calculating the trace from both parts (1.8), we obtain the equation

$$\tau\Big(\frac{\partial \operatorname{tr} S}{\partial t} + \boldsymbol{v}\cdot\nabla \operatorname{tr} S\Big) + \operatorname{tr} S = 2\tau D : S.$$

The last relation is compatible with (1.5) only if $D : S = 0$, which leads to the overdetermination of the equations of motion. The same result is obtained if in equality (1.6), the lower convective derivative is used as $\tilde{d}/dt$. Thus, equalities (1.5) and (1.6) will be compatible only when choosing the Jaumann objective derivative.

Thus, we obtained a closed system of equations (1.1)–(1.3), (1.5), (1.7) for the functions $\boldsymbol{v}$, p, and S. In the general case of three-dimensional motion, this system contains nine quasilinear differential equations of the first order, and in the case of flat plane motion, their number decreases to five.

2. Energy Identity. Below, $\Omega_t \subset \mathbb{R}^3$ denotes the material volume, and Σ_t the surface bounding it. We multiply Eq. (1.2) scalarly by the vector $\boldsymbol{v}$, and in equality (1.6), after division by 2μ, we perform convolution with the tensor S. Combining the obtained relations and integrating the result over the domain Ω_t, we obtain the energy identity

$$\frac{d}{dt}\int\limits_{\Omega_t}\Big(\frac{1}{2}\,\rho|\boldsymbol{v}|^2 + \frac{\tau}{4\mu}\,S : S\Big)\,d\Omega = \int\limits_{\Sigma_t}\boldsymbol{v}\cdot(-p\boldsymbol{n} + S\cdot\boldsymbol{n})\,d\Sigma - \frac{1}{2\mu}\int\limits_{\Omega_t} S : S\,d\Omega. \tag{2.1}$$

Here $\boldsymbol{n}$ is the outward normal unit vector to the surface Σ_t. From relation (2.1), it follows that if Σ_t is a solid impenetrable and immobile surface, it is subject to the no-slip condition

$$\boldsymbol{v} = 0, \qquad x \in \Sigma_t, \quad t > 0; \tag{2.2}$$

if Σ_t is the free boundary, it is subject to the kinematic and dynamic conditions

$$\boldsymbol{v}\cdot\boldsymbol{n} = V_n, \qquad x \in \Sigma_t, \quad t > 0; \tag{2.3}$$

$$-(p - p_0)\boldsymbol{n} + S\cdot\boldsymbol{n} = 2\sigma H\boldsymbol{n}, \qquad x \in \Sigma_t, \quad t > 0. \tag{2.4}$$

Here V_n is the velocity of displacement of the surface Σ_t along the outward normal, p_0 is the atmospheric pressure, H is the mean curvature of the surface Σ_t, and σ is the surface-tension coefficient. If condition (2.2) is satisfied, then the surface integral in identity (2.1) vanishes. Then, we have

$$\frac{d}{dt}\int\limits_{\Omega_t}\Big(\frac{1}{2}\,\rho|\boldsymbol{v}|^2 + \frac{\tau}{4\mu}\,S : S\Big)\,d\Omega = \frac{1}{2\mu}\int\limits_{\Omega_t} S : S\,d\Omega. \tag{2.5}$$

The same occurs if conditions (2.3) and (2.4) are satisfied on this surface. Identity (2.5) implies that, in both cases, the integrand term on its left side decreases monotonically with time and remains unchanged only for $S = 0$. This case corresponds to the motion of the Maxwell medium as a nonderformable solid body.

The integrands on the left sides of equalities (2.1) and (2.5) are nothing but the sum of the kinetic energy of the material volume Ω_t and the energy of elastic shear stresses. If we denote $\mu\tau^{-1} = G$ and identify G with the shear modulus, the expression $(4G)^{-1}S : S$ will coincide with the expression for the shear stress energy density.

We denote the specific internal energy of the medium by U. The total internal energy is the sum of the thermal and elastic energies:

$$\rho U = \rho c T + \frac{\tau}{4\mu}\, S : S \tag{2.6}$$

(T is the absolute temperature and c is the heat capacity of the medium, which is considered constant). Under the assumption that heat transfer in the Maxwell medium obeys the Fourier law with the constant thermal conductivity $\varkappa$, the energy equation can be written as

$$\rho\, \frac{dU}{dt} = \varkappa \Delta T + S : D, \tag{2.7}$$

where d/dt is the convective time derivative. The second term on the right of Eq. (2.7) is a dissipative function.

As noted at the end of Sec. 1, Eqs. (1.1)–(1.3), (1.5), and (1.7) form a closed system for the functions $\boldsymbol{v}$, p, and S. If its solution is known, the temperature of the medium is determined from Eq. (2.7) into which expression (2.6) is substituted. Without considering in detail this question, we note that, in the experiments described in [6, 7], the effect of viscous dissipation on the heating of the medium is negligible due to the smallness of the flow velocities.

Thus, it makes no sense to study temperature effects in the present work. However, the compatibility of the proposed model of an incompressible Maxwell medium with the second law of thermodynamics is to be verified. We write the main thermodynamic identity:

$$\rho\, \frac{dU}{dt} = \rho T\, \frac{ds}{dt} + S : D \tag{2.8}$$

(s is the specific entropy). Relation (2.8) does not contain pressure since in the case of an incompressible medium, it is not a thermodynamic variable. Relations (2.7) and (2.8) lead to the relation

$$\rho T\, \frac{ds}{dt} = \varkappa \Delta T \tag{2.9}$$

(entropy production equation). According to (2.8), elastic strains do not participate in entropy production due to their reversibility.

Let us assume that the material volume Ω_t is heat insulated. Then, by virtue of (2.9), the entropy of the moving volume

$$L(\Omega_t) = \int\limits_{\Omega_t} \rho s\, d\Omega$$

obeys the equation

$$\frac{dL}{dt} = \int\limits_{\Omega_t} \frac{\varkappa}{T}\, |\nabla T|^2\, d\Omega.$$

Thus, the entropy of the isolated material volume does not decrease with time, and its conservation is possible only in the case of isothermal processes. We note that the thermodynamics of deformation of viscoelastic media are considered in great detail in [4, 8].

3. Two-Dimensional Motion of a Maxwell Medium. System (1.1)–(1.3), (1.5), (1.7), which describes the motion of an incompressible viscoelastic Maxwell medium in the general three-dimensional case, is difficult to study. The problem lies not only in its quasilinearity and high order but also in that this system does not have a definite type. Below, we consider two-dimensional motions. We will introduce the following notation:

$$x_1 = x, \quad x_2 = y, \quad v_1 = u, \quad v_2 = v, \quad S_{11} = -S_{22} = A, \quad S_{12} = S_{21} = B.$$

The functions u, v, p, A, and B satisfy the system of equations

$$
\begin{gathered}
u_x + v_y = 0, \\
\rho(u_t + uu_x + vu_y) + p_x - A_x - B_y = 0, \\
\rho(v_t + uv_x + vv_y) + p_y - B_x + A_y = 0, \\
\tau[A_t + uA_x + vA_y + B(v_x - u_y)] - 2\mu u_x + A = 0, \\
\tau[B_t + uB_x + vB_y - A(v_x - u_y)] - \mu(u_y + v_x) + B = 0.
\end{gathered} \tag{3.1}
$$

Let the equality $\varphi(x, y, t) = 0$ specify the characteristic surface of system (3.1). The differential equations of characteristics have the form

$$
\begin{gathered}
\varphi_t + u\varphi_x + v\varphi_y = \pm\Big[\frac{\mu}{\rho\tau}\,(\varphi_x^2 + \varphi_y^2) - \frac{2B}{\rho}\,(\varphi_x\varphi_y)\Big]^{1/2}, \\
\varphi_t + u\varphi_x + v\varphi_y = 0, \qquad \varphi_x = \pm i\varphi_y.
\end{gathered} \tag{3.2}
$$

It is reasonable to draw an analogy between the characteristics of system (3.1) and the characteristics of the system of equations of plane-parallel motion for an ideal incompressible fluid. The latter contains three first-order equations and has two complex characteristics and one trajectory characteristic. The equations of these characteristics coincide with the last three equations in (3.2). The presence of complex characteristics for both systems is due to the incompressibility of the medium. The first two characteristics (3.2) of a viscoelastic media should be called sound characteristics; however, the fact that the inequality $|B| \leqslant \mu\tau^{-1}$, which provides nonnegativity of the radicand in the two first equations (3.1), is satisfied at the initial time does not guarantee that this inequality will be satisfied during motion. If the inequality $|B| > \mu\tau^{-1}$ is satisfied, the development of short-wave Hadamard type instability is possible. Lions and Masmoudi [9] considered a modified model of an incompressible viscoelastic Maxwell medium obtained by addition of the viscous term $\mu\Delta\boldsymbol{v}$ to the right side of the momentum equation (1.2). One of the results of [9] is a proof of the global unique resolvability of the two-dimensional initial-boundary-value problem for the modified system with no-slip conditions on the boundary of the flow domain. The equations of characteristics for the system of equations describing the three-dimensional motion of an incompressible viscoelastic Maxwell medium without the assumption $\operatorname{tr} S = 0$ are given in [10]. In the same paper, the formulation of initial-boundary-value problems for this system is discussed.

4. Analytical Solutions. The specificity of system (3.1) is that it is not evolutionary with respect to pressure, and, therefore, the Cauchy problem with the initial data at $t = 0$ cannot be formulated for this system. However, this system can be resolved for the derivatives of all sought functions for the variable x or the variable y.

Let us consider the following Cauchy problem for system (3.1):

$$
u = u_0(y, t), \quad v = v_0(y, t), \quad p = p_0(y, t), \quad A = A_0(y, t), \quad B = B_0(y, t) \quad \text{at} \quad x = 0. \tag{4.1}
$$

The following statement is valid.

Statement. *Let the functions u_0, v_0, p_0, A_0, and B_0 be analytic in some neighborhood of the point (y_0, t_0) and, in addition, let the inequality $u_0 \neq 0$ be satisfied in this neighborhood. Then, problem (3.1), (4.1) has an analytical solution in some neighborhood of the point (x_0, y_0, t_0). This solution is unique in the class of analytic functions.*

The above statement is a consequence of the Cauchy–Kowalewski theorem.

5. Linear Model of Two-Dimensional Motion. From a physical point of view, a more natural problem than (3.1), (4.1) is the initial-boundary-value problem for system (3.1) with the initial data for the functions u, v, A, and B and no-slip conditions (2.2) on the boundary Σ in the domain $\Omega \subset \mathbb{R}^2$. Having no proofs of the resolvability of this problem for the initial quasilinear system (3.1), we consider its linearized variant.

We introduce the stream function $\psi(x, y, t)$ which satisfies the relations

$$
\psi_y = u, \qquad \psi_x = -v.
$$

Linearizing system (3.1) in the state of rest, we obtain the equation

$$\tau\Delta\psi_{tt} + \Delta\psi_t = \nu\Delta\Delta\psi, \qquad (x,y) \in \Omega, \quad t > 0. \tag{5.1}$$

Here $\nu = \mu\rho^{-1}$ is the kinematic viscosity. The no-slip conditions in terms of the stream function are written as

$$\psi = 0, \qquad \frac{\partial\psi}{\partial n} = 0, \qquad (x,y) \in \Sigma, \tag{5.2}$$

where $\partial/\partial n$ denotes differentiation along the outward normal to the curve Σ. Equation (5.1) is of second order in t and, hence, requires the specification of two initial conditions:

$$\psi = \psi_0(x,y), \qquad \psi_t = \psi_1(x,y), \qquad (x,y) \in \Omega, \quad t = 0. \tag{5.3}$$

The first condition in (5.3) specifies the velocity of points of the medium at the time $t = 0$, and the second condition specifies their accelerations. There is a difference between viscous fluid flow in the Stokes approximation [which corresponds to $\tau = 0$ in Eq. (5.1)] and the motion of a viscoelastic medium. In the latter case, at the initial time, in addition to the velocity field, it is necessary to specify the components of the stress tensor deviator

$$A = A_0(x,y), \qquad B = B_0(x,y), \qquad (x,y) \in \Omega, \quad t = 0. \tag{5.4}$$

Then, the function ψ_1 contained in the second condition (5.3), is found as the solution of the Dirichlet problem for the equation

$$\rho\Delta\psi_1 - 2A_{0,xy} + B_{0,xx} - B_{0,yy} = 0.$$

We denote the cylindrical region $Q_N = \{x, y, t\colon (x,y) \in \Omega, \, t \in (0,N)\}$ by Q_N and formulate the statement of the resolvability of problem (5.1)–(5.3).

We make the following assumptions: 1) the curve Σ belongs to the class C^2; 2) the functions ψ_0 and ψ_1 satisfy the conditions $\Delta\psi_0 \in H^1(\Omega)$ and $\Delta\psi_1 \in L^2(\Omega)$; 3) the conditions $\psi_0 = 0$, $\partial\psi_0/\partial n = 0$, $(x,y) \in \Sigma$ are satisfied.

Statement 1. *For any $N > 0$, there exists a unique solution of problem* (5.1)–(5.3) *such that $\Delta\psi \in H^{1,1}(Q_N)$.*

Proof of this statement does not involve significant difficulties and is not given in the present paper. If the function ψ is found, the functions A and B are determined from the last two equations of system (3.1) after their linearization. Actually, the proof reduces to solving two ordinary differential equations of the first order with the initial conditions (5.4), which include x and y as parameters.

6. Group Properties of System (3.1). The largest group admitted by system (3.1) is calculated in [11]. The basic operators of this group have the form

$$\begin{gathered} X_1 = \partial_t, \qquad X_2 = y\,\partial_x - x\,\partial_y + v\,\partial_u - u\,\partial_v + 2B\,\partial_A - 2A\,\partial_B, \\ X_3 = \alpha(t)\,\partial_x + \dot\alpha(t)\,\partial_u - \rho x\ddot\alpha(t)\,\partial_p, \qquad X_4 = \beta(t)\,\partial_y + \dot\beta(t)\,\partial_v - \rho y\ddot\beta(t)\,\partial_p, \qquad X_5 = \gamma(t)\,\partial_p, \end{gathered} \tag{6.1}$$

where α, β, and γ are arbitrary functions t of the class C^∞; dots denote differentiation with respect to the argument. Thus, the admitted group is infinite-dimensional, and, hence, it is pertinent to speak of a Lie pseudo-group rather than group. The operator X_1 corresponds to translation in time, and the operator X_2 to conformal rotations in the planes (x,y), (u,v), and (A,B). The operators X_3, X_4, and X_5 are specific to the equations of an incompressible continuous medium [12]. The first two of them correspond to transformation to a noninertial coordinate system moving along the x (or y) axis with velocity $\dot\alpha$ (or $\dot\beta$). The presence of the operator X_5 among the basic operators implies that the pressure in system (3.1) is determined to within an arbitrary function of time.

Sequentially setting $\alpha = 1$, $\beta = 1$, $\alpha = t$, and $\beta = t$ in (6.1), we obtain the operators of translation and Galilean translation along the x and y axes:

$$Y_1 = \partial_x, \qquad Y_2 = \partial_y, \qquad Y_3 = t\,\partial_x + \partial_u, \qquad Y_4 = t\,\partial_y + \partial_v. \tag{6.2}$$

The Lie pseudo-group admitted by system (3.1) is a source of its exact solutions, of which the simplest are layered flows.

7. Layered Flows. Below, we consider a solution of system (3.1) which is invariant with respect to the translation group along the y axis. In this solution, all sought functions depend only on x and t. Without loss of generality, we can set $u = 0$. The functions v, A, and B satisfy the closed system of equations

$$\rho v_t = B_x, \qquad \tau(A_t + Bv_x) + A = 0, \qquad \tau(B_t - Av_x) + B = \mu v_x. \tag{7.1}$$

If the solution of system (7.1) is known, the function p is found by means of quadrature from the first equation of system (3.1), in which it is necessary to set $u = 0$ and $B_y = 0$.

Unlike the basic system (3.1), system (7.1) is evolutionary with respect to all sought functions. It is hyperbolic if the inequality $\tau A + \mu > 0$ is satisfied. To study the properties of the solution of system (7.1), we consider the case where the relaxation time τ and the dynamic viscosity μ simultaneously tend to infinity, and $\mu\tau^{-1} = \sigma = \text{const}$. In this case, the solution of system (7.1) is formally expanded in the small parameter τ^{-1}. The equations for the main terms of the expansion (in the former notation) have the form

$$\rho v_t = B_x, \qquad A_t = -B v_x, \qquad B_t = (\sigma + A) v_x. \tag{7.2}$$

System (7.2) has the integral

$$(\sigma + A)^2 + B^2 = F^2(x), \tag{7.3}$$

where F is an arbitrary function x. To simplify further transformations, we set $F = \sigma$ and eliminate the function A from system (7.2) by using (7.3). In addition, in the equations obtained for the functions v and B, we transform to the dimensionless variables x', t', v', and B' using the formulas

$$x = l x', \qquad t = l \rho^{1/2} \sigma^{-1/2} t', \qquad v = \rho^{-1/2} \sigma^{1/2} v', \qquad B = \sigma B',$$

where l is a constant which has the dimension of length. Below, the primes above the variables in the resultant system of equations are omitted:

$$v_t = B_x, \qquad B_t = (1 - B^2)^{1/2} v_x. \tag{7.4}$$

(The case where the extraction of the root gives rise to the minus sign on the right of the second equation in (7.4) is reduced to that considered above by replacing B with $-B$.)

The theory of systems of quasilinear hyperbolic equations, similar to (7.4) is well studied (see, for example, [13]). As is known, these equations are characterized by the formation of strong discontinuities of the solution for any smoothness of the initial data. For system (7.4), discontinuities can be found by studying its solutions of the type of simple waves, i.e., the solutions in which the functions v and B are linked by the functional relation $v = f(B)$. Then, by virtue of (7.4), $f = \pm(1 - B^2)^{-1/4}$, and for the function B, choosing the minus sign, we obtain the equation

$$B_t = -(1 - B^2)^{1/4} B_x.$$

The substitution $B = (1 - w^4)^{1/2}$ reduces this equation to the well-known Hopf equation

$$w_t + w w_x = 0,$$

for which a sufficient condition for the occurrence of a strong discontinuity in the solution of the Cauchy problem is a monotonic decrease of the initial function.

The problem of choosing Hugoniot condition on the strong discontinuity that guarantees uniqueness of the solution of the Cauchy problem remains unsolved. This problem is nontrivial because the initial rheological relation (1.6) does not have the form of the conservation law. Another important question is related to the proof of the existence of the classical solution in the small in time to the Cauchy problem for system (7.1). To prove this, it is sufficient to reduce system (7.1) to symmetric form, but the question of the possibility of rendering this system symmetric also remains unsolved.

8. Exact Solutions of Free-Boundary Problems. It follows from the aforesaid that even in a fixed domain, the problems for system (1.1)–(1.3), (1.5), (1.7) and its two-dimensional analog (3.1) are rather intricate. The situation is further complicated if the surface bounding the material volume is free, i.e., *a priori* unknown. In this case, the significance of exact solutions increases.

A universal tool for constructing exact solutions is provided by group analysis of differential equations [14]. The method proposed in [15] to construct invariant and partially invariant solutions of free-boundary problems for the Navier–Stokes equations is based on the invariance property of the conditions on the free boundary under transformations of some subgroup admitted by the system of Navier–Stokes equations. It turned out that the specificity of this system is of no significance. This allows the proposed approach to be extended to free-boundary problems for the investigated model of an incompressible viscoelastic Maxwell medium.

Let us consider system (3.1), which describes two-dimensional motions of a Maxwell medium and the Lie pseudo-group admitted by this system with the basic operators (6.1). We denote by G_6 the subgroup of this pseudo-group generated by the operators X_1, X_2, Y_1, Y_2, Y_3, and Y_4 [see (6.2)].

Theorem 1. *Let H be an arbitrary subgroup of the group G_6, and let the free surface Σ_t be a nonsingular invariant manifold H. Then, conditions* (2.3) *and* (2.4), *which are satisfied on this surface, are also invariant with respect to H.*

Proof of this theorem is similar to the proof of Theorem 1 in [12, Chapter 6] and is not given in the present paper. There is a natural extension of this theorem to the case of three-dimensional motions of a Maxwell medium with a free boundary. An invariant solution of Eqs. (1.1)–(1.3), (1.5), and (1.7) which describes the process of filling of a spherical hollow in a Maxwell medium under constant pressure at infinity is considered in [16].

It is necessary to note that the formulated theorem allows one to construct not only invariant but also partially invariant solutions and their generalizations. As an example we consider the subgroup H_2 of the group G_6 with the basic operators Y_1 and Y_3. System (3.1) has no solution invariant with respect to H_2 since the rank of the corresponding Jakobi matrix [14] in (3.1) is smaller than the number of the sought functions in this system. The class of partially invariant solutions is nonempty but rather narrow. As shown in [17], the set of exact solutions is considerably extended if one does not require that part of the sought functions be invariant with respect to the group H_2. In [17], it was not assumed that the condition $\operatorname{tr} S = 0$ is satisfied. Below, this condition is considered to be satisfied. Omitting the procedure of constructing the solution, we give the final result which needs to be directly verified.

We consider the problem of symmetric deformation of a viscoelastic strip with rectilinear free boundaries which are subject to the kinematic condition (2.3) and the condition of equality of the normal pressure to atmospheric pressure [one of conditions (2.4)]. The tangential stresses on the boundaries of the strip are linearly distributed. The flow domain is $\Omega_t = \{x, y\colon\ x \in \mathbb{R},\ |y| < l(t)\}$, and the lines $y = \pm l(t)$ are free boundaries. The solution is represented as

$$u = xh(t), \qquad v = -yh(t), \qquad p = r(y,t), \qquad A = q(y,t), \qquad B = Cxy \exp(-t/\tau).$$

Here C is a specified constant and the function $h(t)$ is a solution of the Cauchy problem

$$\rho(h' + h^2) = C \exp(-t/\tau), \quad t > 0, \qquad h(0) = h_0.$$

The expression for $h(t)$ in terms of Bessel functions is given in [17]. The initial width of the strip is specified and equal to $2l_0$. The specification of the constant h_0 determines the initial velocity field, which turns out to be linear. This solution is a generalization of the well-known Ovsyannikov solution [18] which describes the deformation of a strip of an ideal incompressible fluid with free boundaries. The initial stress field is given by the formulas

$$A = q_0(y), \qquad B = Cxy,$$

where $q_0(y)$ is a specified function. If the function $h(t)$ is known, $q_0(y)$ is determined by solving the linear problem for a second-order parabolic equation. The pressure $r(y,t)$ is determined by means of quadrature. Finally, the function $l(t)$ (strip half-width) is represented as

$$l(t) = l_0 \exp\Big(-\int\limits_0^t h(z)\,dz\Big).$$

In the obtained solution, the strip length is not limited. For large values of x, the condition of absence of complex sound characteristics $\tau|B| < \mu$ for this solution is violated. This implies the development of short-wave instability of the solution at large distances from the line $x = 0$. However, it is possible to consider the restriction of this solution to the domain $\omega_t = \{x, y\colon\ |x| < b(t),\ |y| < l(t)\}$, where $b(t)$ is a specified function. On the specified boundaries of the domain ω_t, it is necessary to provide satisfaction of the boundary conditions compatible with the form of the exact solution. If the initial length of the strip $2b_0$ is small enough so that $C\tau b_0 l_0 < \mu$, the Hadamard instability will be suppressed.

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